OAP20

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Calculation Sheet

Job No.	Sheet No.
	Member/Location
Job Title ILLUMINATION BY MOONLIGHT	Drg. Ref.
	Made by / Date / Chd.

DIAMETER OF MOON = 3476 km

MEAN DISTANCE = 384,400 km

AREA OF MOON DISC = $\pi \times \left(\left(\frac{3476}{2}\right) \times 10^3 \right)^2$

$= 9.49 \times 10^{12}\ m^2$

LUMINANCE OF MOON $\approx 4 \times 10^3\ cd/m^2$ ref (1)

$\therefore$ LUMINOUS INTENSITY = Area × luminance

$= 3.8 \times 10^{16}$ candelas.

$\therefore$ ILLUMINATION AT TOP OF EARTHS ATMOSPHERE = luminous intensity / distance^2

= 0.26 lux.

(1) Henderson - Daylight + its spectra

THE SITE

Latitude	:	43° 44' North of Equator	
Longitude	:	3° 07' East of Greenwich	
Maximum altitude of full moon	:	June	19°
		May and July	29°
		April and August	40°
Maximum azimuthal movement of moon in one hour	:	18 degrees	
Maximum altitudinal movement of moon in one hour	:	9 degrees	
End of astronomical twilight	:	June 21st	22:52
		July 21st	22:16
		August 21st	21:20

THE MOON

Physical Data

Mean distance	:	384,000 km
Mean diameter	:	3476 km
Apparent angular size at mean distance	:	31' 05"
Eccentricity of orbit	:	0.0549
Distance at perigee	:	356,410 km
Distance at apogee	:	406,740 km
Plane of orbit	:	5 ° 9' to ecliptic

Photometric Data

Albedo (reflectance)	:	0.07
Magnitude of full moon	:	$-12^{m}.5$
Luminance of full moon	:	4,000 cd/m^2
Luminous intensity of full moon	:	3.8×10^{16} candelas
Extraterrestial illuminance from full moon	:	0.26 lux
Illuminance at sea level, normal to full moons rays	:	0.19 lux

AREAS OF FURTHER STUDY

Visual Acuity and Perception of Detail

1. Target level of perception
2. Role of colour and contrast in perception
3. Required illumination levels on stage generally
4. Adaption of eye to low light levels

Performance Area

1. Size of performance area
2. Distance and position of audience relative to stage

Collector Design

1. Number and area of collector(s) required
2. Position of collectors relative to stage
3. Shape and orientation of collectors
4. Tolerance of manufacture
5. Rigidity and deflections

Luminous Efficacy of Light Sources

The luminous efficacy of a light source is defined as the ratio of the total luminous flux (lumens) to the total power input (watts or equivalent).

The maximum luminous efficacy of an "ideal" white source, defined as a radiator with constant output over the visible spectrum and no radiation in other parts of the spectrum is approximately 220 lumens per watt.

GENERATION OF LIGHT

Natural Phenomena

Sunlight. Energy of color temperature about 6500 K is received from the sun at the outside of the earth's atmosphere at an average rate of about 1350 watts per square meter.[4] About 75 per cent of this energy reaches the earth's surface at sea level (equator) on a clear day.

The average luminance of the sun is approximately 1600 megacandelas per square meter viewed from sea level. The illuminance on the earth's surface by the sun may exceed 100 kilolux [10,000 footcandles]; on cloudy days the illuminance drops to less than 10 kilolux [1000 footcandles]. See Section 7.

Sky Light. A considerable amount of light is scattered in all directions by the earth's atmosphere. The investigations of Rayleigh first showed that this was a true scattering effect. On theoretical grounds the scattering should vary inversely with the fourth power of the wavelength when the size of the scattering particles is small compared to the wavelength of light, as in the case of the air molecules themselves. The blue color of a clear sky and the reddish appearance of the rising or setting sun are common examples of this scattering effect. If the scattering particles are of appreciable size (the water droplets in a cloud, for example), scattering is essentially the same for all wavelengths. (Clouds appear white.) Polarization in parts of the sky may amount to 50 per cent.

Moonlight. The moon shines purely by virtue of its ability to reflect sunlight. Since the reflectance of its surface is rather low, its luminance is approximately 2500 candelas per square meter. Illumination of the earth's surface by the moon may be as high as 0.1 lux [0.01 footcandle].

Lightning. Lightning is a meterological phenomenon arising from the accumulation, in the formation of clouds, of tremendous electrical charges, usually positive, which are suddenly released in a spark type of discharge. The lightning spectrum corresponds closely to that of an ordinary spark in air, consisting principally of nitrogen bands, though hydrogen lines may sometimes appear owing to dissociation of water vapor.

Aurora Borealis (Northern Lights) and Aurora Australis (Southern Lights). These hazy horizontal patches or bands of greenish light on which white, pink, or red streamers sometimes are superposed appear 60 to 120 miles above the earth. They are caused by electron streams spiraling into the atmosphere, primarily at polar latitudes. Some of the lines in their spectrum have been identified with transitions of valence electrons from metastable states of oxygen and nitrogen atoms.

Bioluminescence. "Living light" is a form of chemiluminescence (see page 2-13) in which special compounds manufactured by plants and animals are oxidized, producing light. Although it has been proven that oxygen is required to produce bioluminescence, there is no evidence that the light producing compound must be a "living" material. The light producing compound can be dried and stored many years and upon exposure to oxygen emit light.

"Man-Made" Sources

Historically, light sources have been divided into two types—incandescent and luminescent. Fundamentally, the cause of light emission is the same, *i.e.*, electronic transitions from higher to lower energy states. The mode of electron excitation is different, however, as well as the spectral distribution of radiation. Incandescent solid substances basically emit a continuous spectrum while gaseous discharges radiate mainly in discrete spectral lines; however, there is some overlapping. Incandescent rare earth elements can emit lines, whereas high pressure discharges produce a continuous spectrum.

The two classical types, with subdivisions showing associated devices or processes, are listed as follows (see Section 8):

1. Incandescence
   - Filament lamps
   - Pyroluminescence (flames)

ological ages. Strong evidence exists that the ocean and the atmosphere are secondary products of accretion—that is, have been slowly formed in the course of the Earth's history from material coming out of the rocks.

A more definitive theory of mountain building arose in the 1960s and 1970s. The discovery of the extent and nature of the Earth's mid-oceanic ridges and the fact that sea floor materials became older along paths normal to the ridges led to the concept of sea-floor spreading—that new ocean floor is generated by the upwelling of the mantle along the ridge crests, that it subsequently moves laterally away from the ridges and toward the continents, and that it is eventually subducted or dragged downward back into the mantle along continental margins. These processes are thought to result in both continental drift and mountain building and are further discussed in the article CONTINENTAL LANDFORMS.

(W.M.E.)

The Moon

The Moon, designated ☾ in astronomy, is a natural satellite of the Earth, but it is sometimes described as a planet, because it is larger than most known natural, planetary satellites and, because of its size, has many of the properties of a small planet such as Mercury. The Moon, like the Earth and the other terrestrial planets, has a solid body, nearly spherical in shape. The Moon's radius is about 1,738 kilometres, only 0.2725 that of the equatorial radius of the Earth; this difference, with the fact that the Moon's mass is only $1/_{81.3}$ of the Earth's mass, is largely responsible for the difference between the atmospheres of the two bodies. The escape velocity of a particle from the surface of the Moon is about 0.213 that from the Earth, and, if past temperature conditions were much like present ones, the gas molecules and atoms of any early lunar atmosphere would have escaped in a time that is short when compared with the Moon's lifetime.

The Moon's mean density is about 3.34 grams per cubic centimetre, close to the density of the Earth's mantle; but studies of rocks returned from the Moon's surface indicate that the Moon's interior composition is not the same as that of the Earth's mantle and that in the Moon there is only a very small (possibly no) core. Many of the rock samples retrieved from the large, relatively smooth, dark areas of the face of the Moon called the maria are best described as basalts, but they are quite uncharacteristic of any known terrestrial rock or meteorite. They are about the colour of graphite—charcoal gray.

The whole Moon reflects less than one-tenth of the light that falls on it. The low reflecting power of the Moon had been known for a long time from estimates of its overall albedo. "Albedo" is a Latin word meaning "whiteness." While it is relatively easy to measure the relative brightness, or reflecting power (normal albedo), of flat surfaces illuminated and viewed normally in the laboratory, it is more difficult to assess the overall albedo of a spherical planet. The brightness of the Moon varies through its phases because of the roughness of its surface and the resulting variable amount of shadow. The Full Moon, when the Sun, Earth, and Moon are in the same line, is 10 times as bright as the Moon at First Quarter, the position of the satellite when it has completed one-quarter of its orbit around the Earth. When three-quarters of the orbit is completed, the Moon is said to be in the Last Quarter.

Variable brightness

(G.Fi.)

THE MOTION OF THE MOON

The apparent motion of the Moon. The lunar orbit is nearly circular, but it is more accurately described as an ellipse with the Earth at one focus, its eccentricity being about 0.0549. As seen from the Sun, however, the Moon sweeps out a sinuous concave path because its elliptic geocentric motion is superimposed on the elliptical orbit motion of the centre of gravity of the Earth–Moon system about the Sun.

The month. The Moon's motion around the Earth may be judged by reference to the Sun, the stars, the Earth's equator, or the Earth's orbit. To an Earth-bound observer, it is the fastest moving permanent natural object in the sky, making a complete circuit of the Earth with respect to the stellar background in only 27.322 days. This is the sidereal month.

By the time the Moon has completed one revolution of the Earth relative to the stars, the Earth–Moon system has itself moved through 7.5 percent of its orbit about the Sun. This causes the time between successive Full Moons or successive New Moons (called syzygies) to be longer than a sidereal month. This period of the lunar phases of 29.531 days is called the synodic, or lunar, month. The synodic month is determined by using the Earth–Sun direction as a reference.

If the orbital plane of the Moon were in the ecliptic (the plane of the Earth's orbit around the Sun), there would be a lunar eclipse at every Full Moon and a solar eclipse at every New Moon. Eclipses do not happen so frequently, however, because the lunar orbit is inclined to the ecliptic by 5°.15. The points at which the Moon crosses the ecliptic into the northern and southern celestial hemispheres are called the ascending and descending nodes, respectively. The interval between successive crossings of a given node, called the draconic, or nodical, month, is 27.212 days; the term draconic refers to the dragon that the ancients believed swallowed the Moon and Sun to cause eclipses. The draconic month differs from the sidereal month because the nodes regress (move opposite to the orbital motion), completing a full revolution every 18.6 years. This regression is caused principally by the gravitational pull of the Earth's equatorial bulge. The cycle of lunar and solar eclipses repeats every 18.6 years, a period that is sometimes called the saros.

The mean distance between the Earth and the Moon is 384,400 kilometres, but, because of the ellipticity of the lunar orbit, the actual distance varies about 11 percent, between 363,000 and 406,000 kilometres. At its closest approach to the Earth, a satellite is said to be at perigee, and it is at apogee at its most distant separation; the time interval between successive perigee passages by the Moon of 27.555 days is called the anomalistic month.

Viewed from the celestial north pole, the Moon's geocentric orbital motion, the Earth–Moon's heliocentric motion, and the axial rotations of both Earth and Moon are all counterclockwise. Because the Earth spins west to east in a time that is short compared to the month, the Moon, like the Sun and the stars, appears to rise in the east and set in the west, even though its true motion relative to the celestial north pole is from west to east. Because of its orbital motion, the Moon rises, on the average, about 50 minutes later each day.

Optical librations. Galileo discovered that the Moon always turns the same face to Earth. This means that the Moon rotates on its axis with exactly the same period as its orbital motion, a common feature among natural satellites. While its axial rotation is nearly uniform, the Moon's orbital speed varies by 11 percent—the same amount of variation as in the Earth–Moon distance. The Moon moves faster near perigee and slower near apogee, with an average speed of 1.02 kilometres per second. Thus, from Earth, the Moon seems to oscillate as the orbital speed loses or gains on the constant rotation. This causes features in the region of the limb, the visible edge of the Moon, to slip alternately into and out of view. A patient observer on Earth can eventually see about 57 percent of the total lunar surface. These apparent oscillations of the Moon are known as optical librations and can amount to Moon-centred angular displacements of 8° in longitude and 6°.8 in latitude. The laws governing them were discovered empirically by Giovanni Domenico Cassini in 1693.

The actual motion of the Moon. The mathematical treatment of the Moon's actual geocentric motion is called the lunar theory. Customary astronomical terminology distinguishes between an analytic procedure, called a theory of general perturbations, and a numerical method, called a theory of special perturbations. Because of the Moon's nearness to Earth and of its rapid angular motion, the lunar theory has long played a major role in formulating and testing dynamical theory.

Pre-Newtonian history. In ancient times, the Moon's phenomena were of great religious importance. Thus, considerable attention was paid to its motions; and the observations made were of an exactitude unsought in recording terrestrial phenomena. The Babylonians devised numerical methods for predicting eclipses and heliacal risings. The Greek astronomer Hipparchus in the 2nd century BC discovered the eccentricity of the lunar orbit, the motions of perigee and node, and the inclination of the orbital plane to the ecliptic. He explained the motion as being uniform in a circular orbit with the Earth placed eccentrically, and this was adequate to represent the observations then available. The numerical values of the parameters were derived from eclipse observations, some dating to Chaldean times. Although his results were superior to previous attempts, the restriction to eclipse data led Hipparchus to an incorrect value of eccentricity, through inability to distinguish the major inequality (departure from uniform motion, true value 6°.29), due to eccentricity, from the greatest solar perturbation. The latter inequality, called evection (1°.27), was discovered by the Greek astronomer Ptolemy in the 2nd century AD. Little significant improvement in the knowledge of the lunar motion was secured until the 16th-century Danish astronomer Tycho Brahe discovered the other great inequality caused by solar attraction, called the variation. This phenomenon, whose amplitude is 0°.66, vanishes when the Moon is new, full, or in quadrature. The physical explanation of these effects required the discovery of universal gravitation.

Analytic theory. The analytic treatment of the lunar motion is carried out in two stages. In what is termed the "main problem," it is assumed that the Sun, Earth, and Moon behave as point masses and that the barycentre, or centre of gravity, of the Earth–Moon system describes a fixed ellipse around the Sun. After the solution of this problem, the remaining effects are introduced as small corrections to the main solution. These effects include the gravitational attractions of the planets acting on both the Earth and Moon, the forces caused by the irregular shapes of the Earth and Moon, corrections because of the use of a non-inertial coordinate system, and relativistic corrections.

The solution of the main problem gives the coordinates as harmonic series in which the arguments are linear combinations of the time-variant angles l, l', F, and D. The periods of revolution of l, F, and D are the anomalistic, draconic, and synodic months. That for l' is the anomalistic year in the Earth's heliocentric motion. The series coefficients are power series in these parameters: e ($= 0.055$) the eccentricity and γ ($= 0.045$) the sine of half the inclination of the lunar orbit; m ($= 0.075$) the ratio of average angular speeds of Earth and Moon; a ($= 0.0025$) the ratio of average geocentric distances of Moon and Sun; and e' ($= 0.017$) the eccentricity of Earth's orbit. The values of these parameters are determined from observation. The perturbations developed in the second stage of the analysis introduce the planetary orbit periods and the motion of the Moon's node into the arguments, as well as secular effects that grow without bound with time.

Isaac Newton based his theory of gravitation largely on the comparison of the motion of the Moon with that of bodies falling freely near Earth. He succeeded in proving that the principal periodic inequalities, as well as the motions of node and perigee, were caused by solar attraction. He deduced many new inequalities not previously known, but his motion of perigee was only about half the observed value; this disparity was later reduced to about 8 percent. Newton's theory was published in geometrical form in his *Principia*, but the method of fluxions (developed later into the integral calculus) was probably used in the actual derivation. No substantial advance was made beyond Newton's work until the mid-18th century, when the lunar motion was studied by many mathematicians.

Outstanding lunar theories were those obtained in the 19th century by Peter Andreas Hansen, Charles Eugène Delaunay, and George William Hill and Ernest William Brown. Hansen's theory was the most nearly numerical of the analytic theories, and it was of greater accuracy than any other before Brown; it was used worldwide from 1862 to 1922. Corrections by Simon Newcomb were applied after 1883. Hansen's work has served as the basis for orbit computations for artificial satellites.

The most complete algebraic development of the lunar theory until the present time was that of Delaunay. It was never adequate for comparison with observations, but it has been valuable in theoretical studies, the extension of other lunar theories, and the analysis of the orbits of natural satellites of other planets.

Hill developed methods that allowed the achievement of results of given accuracy with much reduced labour. These methods have been effective in studies of the general problem of three bodies (see MECHANICS) as well as in the practical solution of the lunar problem. The two principal features were (1) the introduction of the value of m from the outset to avoid convergence difficulties and (2) the use of a rectangular coordinate system rotating at the mean angular rate of the Moon. Hill's elegant method for determining the motion of perigee was later applied by John Couch Adams to the motion of the node. Based largely on Hill's work, Brown then developed the lunar theory that has been generally used to compute the coordinates of the Moon since 1923, although it was still necessary to replace the theoretical motions of node and perigee with the observed values, as the former were unsatisfactory.

Brown's series contain about 1,500 terms, five times as many as in Hansen's theory. Reduced to tables, it filled 650 quarto pages, and one man working alone with a desk calculator could extract the coordinates just fast enough to keep up with the real Moon. The electronic computer relieved this burden by permitting, since 1952, the direct calculation of the series. At the same time, corrections were introduced to compensate for the variations in the Earth's rotation rate and for the effect of tidal friction.

Brown's theory remained adequate for even the most precise astronomical uses for nearly half a century, and was still used for national almanacs in 1981. During the 1960s, the technologies of space exploration and radio astronomy, however, introduced the need for an enormous improvement in both the accuracy and internal coherence of the theory. At the same time, the rapid development of electronic computer systems for algebraic manipulation brought a resurgence of interest in the construction of analytic theories. The powerful new mathematical technique of Lie series, applied by computer, has permitted a return to the purely algebraic approach abandoned after Delaunay. Of the efforts to produce radically new theories, none had been sufficiently completed for general use by 1981. Along with the lunar coordinates, some of these efforts provide the necessary functions for correcting the lunar orbit parameters, an important innovation.

Numerical lunar theory. Although strictly numerical orbit theories have long been constructed for some solar system objects, this was impractical for the lunar motion before the development of the electronic computer. The nearest approach to a special theory was the attempt by Sir George Biddell Airy in 1883. He was unsuccessful and the idea was abandoned.

Space exploration required more accurate lunar positions than could be obtained from extant analytic theories; the solution was to compute the ephemerides by numerical integration of the equations of lunar orbital motion. This technique had been used for several decades for planetary orbits, but its application to the lunar problem was more difficult because of the Moon's high angular speed and the accuracy with which small perturbations can be observed. Numerical ephemerides first came into general use in 1969 and are now widely used for the most exacting applications, such as spacecraft navigation, lunar laser ranging, and radio interferometry. The method of special perturbations has the advantages of arbitrarily high internal precision and simultaneous inclusion of all desired perturbing forces in one basic calculation. Since both laser and radio observations can detect motions of a few centimetres, the calculations must account for such effects as planetary attractions, the Earth's oblateness, lunar libration, tidal friction in the Earth, and very complicated relativistic effects. The coupling between orbital and lunar rotation is now observable, so simultaneous numerical integration of these two motions is more commonplace.

Use of numerical ephemerides

The major disadvantage of the numerical approach is the uncorrectibility of a special theory; improvements to the parameter values require a completely new integration.

Cassini's laws for lunar rotation

The rotation of the Moon. Cassini's laws for the rotation of the Moon go beyond the fact of synchronous spin. They state, in addition, that the equator of the Moon is inclined by about 1°.5 to the ecliptic (6°.7 to the lunar orbit), and that the rotation axis always lies in a plane perpendicular to the line of nodes of the lunar orbit on the ecliptic. Thus, the axis precesses in space with the same period of 18.6 years as the nodes. The synchronism and the axial precession are evidence of a gravitational resonance in which the Earth controls the Moon's spin. The apparent monthly oscillations that result are due primarily to the parallax of an Earth-bound observer. The Moon does oscillate in its rotation, but these wobbles, called librations, are so small as to be undetectable except by precise observation.

Physical librations. The Moon is not spherically symmetric (see below, *The mass and gravity of the Moon*). The gravitational attraction of the Earth on the excess mass in the irregularities of the Moon's shape produces torques that influence rotation. Even the Cassini motion is caused by a bulge on the Moon along the principal axis pointed in the mean direction to Earth. The Earth's pull tends to bring its satellite back into line following any movement (caused by variations in the Moon's orbital speed) away from the true Earth direction. Consequently, the actual rotation forced by gravitational attractions includes a complex pattern of small motions that can exceed 100 arc seconds in selenocentric (Moon-centred) rotation. These motions are called physical librations. Since they arise entirely from gravitational action, they can be calculated from mathematical theory, following principles laid down by Sir Isaac Newton and Leonhard Euler. Such theories have only arisen in the past century, owing to the difficulties of observation; they are now constructed both by analytic methods and by numerical integration, similarly as (and sometimes simultaneously with) the lunar orbit theories.

Free librations. The lunar surface shows evidence of many impacts by external bodies, both large and small. In any such impact, much of the energy of the impacting object is transmitted to the Moon. Some of that energy produces yet another rotational motion, a type of Eulerian oscillation that, in specific reference to the Moon, is called free libration. The mathematical theory of rotating bodies specifies that these free librations will occur at only three frequencies, the periods of which are predictable from the physical properties of the Moon. Since these librations depend on the size and direction of the impacting object, however, their amplitudes cannot be predicted and can be determined only from observation. Once stimulated by an impact, the free librations gradually disappear by internal friction, and so are expected to be very small. The first claim of detection of a free libration was made in 1967, but this now seems spurious; modern data appear to show a free libration in longitude, but it cannot be larger than a selenocentric rotation of 2 arc seconds, 0.01 arc second as seen from the Earth.

Interpretation of small irregularities in the motion. The lunar motion provided Newton with the key to universal gravitation, and it has remained for three centuries one of the most difficult and fruitful practical problems in celestial mechanics. In conjunction with new methods of observation of unprecedented accuracy, recent work has confirmed the classical value of tidal friction, given evidence of a dense lunar core, determined the location and motion of the equinox, invalidated the Brans-Dicke scalar-tensor relativity theory, suggested a weakening of gravity, and improved knowledge of the shape, mass, and gravity field of the Moon. In the 1970s, the accuracy of the theories of both the orbit and rotation of the Moon was improved by a factor of nearly 100.

Such discoveries were made possible by the analysis of small irregularities in the observed motion of the Moon. For each observation, the mathematical theory is used to compute the place at which the Moon (or, more correctly, the observed point on the Moon) should have been if all aspects of the theory were correct. When this computation is compared with the observation, there remains a small difference, called a residual. When residuals from many observations are compared, systematic patterns appear that are clues to errors or omissions in the mathematical theory. Each contributing phenomenon (*e.g.*, mass, gravity, or observer's location) has a characteristic "signature" in the residuals. Improvement of the mathematical theory, or the discovery of new phenomena, is then derived from an analysis of the residuals to recognize and explain various signatures, many of which are present in the same data. It is essentially a problem in mathematical decoding. For example, in the 19th century, gravitational theory could not explain all of the motion of the Moon then observed. At first, this failure was thought to be due to a deficiency in the orbit theory. It was later shown to be caused, however, by previously unknown fluctuations in the Earth's rate of rotation. These fluctuations affect the apparent motion of all celestial objects, but the magnitude of the effect is proportional to the object's orbital speed, and so it was first discovered in the lunar motion. From 1954 to 1976, observations of the Moon were used to refine the determination of time from the rotation of Earth; more precise refinements are now achieved by the use of atomic clocks.

Using new techniques of occultations, radio interferometry and laser ranging, much finer details of the lunar orbit and rotation are now observable than was formerly possible. Such data are, in fact, so precise that it is no longer feasible to consider the orbital and rotational motions of the Moon as unrelated subjects.

An occultation is the disappearance or reappearance of a star from the view of an Earth-based observer as the Moon passes in front of it. In a normal occultation, there is one disappearance and one reappearance in a given event. In a grazing occultation, however, the edge of the Moon's disk passes in front of the star so that the star disappears behind each of a series of lunar mountains and reappears in each of the valleys between them. In both cases, measurements of the lunar surface can be made; these measurements can be accurate within a few metres at the Moon, but their further refinement is limited by the Moon's topography. Occultations have been used primarily to study the shape of the Moon and the long-term evolution of its orbit, including the effects of both tidal friction and the possible weakening of gravity.

Very long baseline (radio) interferometry (VLBI) is used to determine the direction of a celestial source of radio waves. It measures the parallax effect of two simultaneous observations of radio waves by Earth-based stations separated by large distances. The Moon is not a natural source of radio waves, and so radio transmitters were landed on the Moon by various spacecraft to send such radio signals back to Earth. Another technique of lunar observation is that of lunar laser ranging (LLR), which provides measurements of the Moon's distance from Earth. Like VLBI, LLR required the placement of man-made devices—in this case, passive reflectors, or mirrors—on the lunar surface. In LLR, a short light-pulse is emitted through an Earth-based telescope to the Moon, where it is reflected from one of the mirrors back to Earth, where the light travel-time is measured. Although LLR observations have been made at several places, more than 95 percent of them have been made at McDonald Observatory, where an LLR program has been in operation since the landing of Apollo 11 on the Moon in 1969.

Both VLBI and LLR provide data on the position of the Moon that is accurate within a few centimetres. Using these techniques, important discoveries have been made since 1969. In the question of gravitational theory, for example, the lunar motion continues in its important role; new data have shown the so-called Nordtvedt effect, a violation of the principle of equivalence, to be essentially zero, thereby eliminating the Brans-Dicke-Jordan theories of relativity as serious alternatives to Einstein's general relativity theory. No convincing evidence has yet been found for a variation of the gravitational constant, a variation required by some theories of relativity. In other results, terrestrial tidal friction has been found to affect the lunar cial Perturbation Theory for the Lunar Motion," in *Astron. Jour.*, 75:1133–39 (1970), the first published use of a numerical lunar orbit, showing the inadequacy of Brown's theory. MARTIN A. SLADE, *The Orbit of the Moon* (1971), including mathematical details of numerical integration of the lunar motion. J. DERRAL MULHOLLAND, "Scientific Achievements from Ten Years of Lunar Laser Ranging," in *Rev. Geophys. Space Phys.*, 18:549–564 (1980) and "How High the Moon. A Decade of Laser Ranging," in *Sky & Telescope*, 60:274–279 (October 1980). J. DERRAL MULHOLLAND, "The Rotation of the Moon," in *Bull. Cl. Sciences Acad. Roy. Belgique*, Ser. V, 55:1088–1113 (1974), a comprehensive overview. STANTON PEALE, "Generalized Cassini's Laws," in *Astron. Jour.*, 74:483–489 (1969), an analysis of the resonance. DONALD H. ECKHARDT, "Computer Solutions of the Forced Physical Librations of the Moon," in *Astron. Jour.*, 70:466–471 (1965), a development of the semi-analytic solution; A. MIGUS, *Theorie analytique de la libration physique de la lune* (1977), a development of the analytic theory; and ROGER J. CAPPALLO, *The Rotation of the Moon* (1980), which applies numerical integration to the Moon's libration. The use of different kinds of observations is described in R.W. KING, C.C. COUNSELMANN III, and IRWIN I. SHAPIRO, "Lunar Dynamics and Selenodesy: Results from Analysis of VLBI and Laser Data," in *Jour. Geophys. Res.*, 81:6251–56 (1976); J. DERRAL MULHOLLAND (1980, *op. cit.*); and LESLIE V. MORRISON, "An analysis of Lunar Occultations in the Years 1943–1974 for Corrections to the Constants in Brown's Lunar Theory," in *Mon. Not. Roy. Astron. Soc.*, 187:41–82 (1979). THOMAS C. VAN FLANDERN, "Is Gravity Getting Weaker?" *Scient. Amer.*, 234: 44–52 (February 1976), an explanation of the effect of gravity on the Moon's orbit. The mass of the Moon is discussed in A.J. FERRARI *et al.*, "Geophysical Parameters of the Lunar Orbit," in *Jour. Geophys. Res.*, 85: 3939–51 (1980). For the gravity field of the Moon, see WILLIAM M. KAULA, *An Introduction to Planetary Physics* (1968); P.M. MULLER and W.L. SJOGREN, "Mascons: Lunar Mass Concentrations," in *Science*, 161:680–684 (1968), which announced the discovery of mascons; and A.J. FERRARI *et al.*, *op. cit.*, which gives the most complete description of the harmonic field to date.

Types of lunar features are covered in GILBERT FIELDER, *Lunar Geology* (1965); ELBERT A. KING, *Space Geology* (1976); JOSIAH E. SPURR, *Geology Applied to Selenology*: vol. 1, *The Imbrium Plain Region of the Moon*, vol. 2, *Features of the Moon*, vol. 3, *Lunar Catastrophic History*, and vol. 4, *The Shrunken Moon* (1944–49); GILBERT FIELDER and LIONEL WILSON (eds.), *Volcanoes of the Earth, Moon, and Mars* (1975); NICHOLAS M. SHORT, *Planetary Geology* (1975); THOMAS A. MUTCH, *Geology of the Moon* (1970); PETER H. SCHULTZ, *Moon Morphology* (1976); ROYAL SOCIETY OF LONDON, *The Moon* (1977); G.P. KUIPER (ed.), *Photographic Lunar Atlas* (1960, with suppl., *Orthographic Atlas of the Moon*); E.A. WHITAKER *et al.*, *Rectified Lunar Atlas* (1963); *The Times Atlas of the Moon* (1969); J.E. BALDWIN, "Thermal Radiation from the Moon and the Heat Flow Through the Lunar Surface," *Mon. Not. R. Astr. Soc.*, 122:513–522 (1961); ROBERT HOOKE, *Micrographia* (1st ed., 1665; 2nd issue, 1667, reprinted 1938); JAMES NASMYTH and JAMES CARPENTER, *The Moon*, 4th ed. (1903); WILLIAM H. PICKERING, "Lunar and Hawaiian Physical Features Compared," *Mem. Am. Acad. Arts Sci.*, 13:151–182 (1908); J.E. SPURR (*op. cit.*); G.K. GILBERT, "The Moon's Face," *Bull. Phil. Soc. Wash.*, 12:241–292 (1893); RALPH B. BALDWIN, *The Measure of the Moon* (1963) and *A Fundamental Study of the Moon* (1965); BRIAN MASON and WILLIAM G. MELSON, *The Lunar Rocks* (1970); G.M. BROWN, "Geochemistry of the Moon," *Endeavour*, 30:147–151 (1971); JOHN F. LINDSAY, *Lunar Stratigraphy and Sedimentology* (1976); ALFRED A. LEVINSON and S. ROSS TAYLOR, *Moon Rocks and Minerals* (1971); TYPHOON LEE, "New Isotopic Clues to Solar System Formation," in *Rev. Geophys. and Space Phys.*, 17:1591–1611 (1979); ALAN H. COOK, *Physics of the Earth and Planets* (1973); S. ROSS TAYLOR, *Lunar Science: A Post-Apollo View* (1975). The results of intensive studies of returned lunar rocks have been published in *Science*, vol. 167, no. 3918 (1970); in the reports of the Lunar Science Conferences held since 1970; and by BRIAN MASON and WILLIAM G. MELSON (*op. cit.*). Discussions on the evolution of the Moon's orbit are found in SIR GEORGE H. DARWIN, *The Tides and Kindred Phenomena in the Solar System* (1898, reprinted 1962), the classic first work on the subject; W.H. MUNK and G.J.F. MACDONALD, *The Rotation of the Earth* (1960), a landmark in the study of dissipation in the solar system; G.D. ROSENBERG and S.K. RUNCORN (eds.), *Growth Rhythms and the History of the Earth's Rotation* (1975), a good summary of the use of ancient fossil records, but with obsolete astronomical discussions; KURT LAMBECK, *The Earth's Variable Rotation* (1980). Theories of lunar origin are dealt with by BRIAN G. MARSDEN and A.G.W. CAMERON (eds.), *The Earth-Moon System* (1966); S. FRED SINGER, "The Origin of the Moon and Geophysical Consequences," in *Geophys. Jour. Roy. Astron. Soc.*, 15:205–226 (1968); WILLIAM M. KAULA and A.W. HARRIS, "Dynamics of Lunar Origin and Orbit Evolution," in *Rev. Geophys. Space Phys.*, 13:363–372 (1975).

(J.D.Mu./G.Fi.)

Mars. Early ideas on the possibility of life on Mars are summarized in H. STRUGHOLD, "Synopsis of Martian Life Theories," in *Advances in Space Science and Technology*, 9:105–127 (1967); post-Viking views are discussed in HAROLD P. KLEIN, "The Viking Mission and the Search for Life on Mars," in *Reviews of Geophysics and Space Physics*, 17:1655–62 (October 1979). The nature of Martian phenomena as understood before 1969, including several enlightening chapters on the possibilities of extraterrestrial life, are discussed in SAMUEL GLASSTONE, *The Book of Mars*, NASA SP-179 (1968). The era of space exploration is covered by several books and collections of papers, each dedicated to an exposition of the subject at the conclusion of a project. Mariners 6 and 7 are discussed in N.W. CUNNINGHAM and H.M. SCHURMEIER, *Mariner Mars 1969: Preliminary Report*, NASA SP-225 (1969). An illustrated summary of the results of the Mariner 9 orbiter mission is to be found in *Mars as Viewed by Mariner 9*, NASA SP-329 (1974). In addition, *Icarus*, 17:289–561 (October 1972) is dedicated to a collection of original papers by Mariner 9 experimenters. A synthesis of views on the nature of the planet between Mariner 9 and Viking is developed in THOMAS A. MUTCH *et al.*, *The Geology of Mars* (1976). The Viking results are extensive; the original papers describing the results of the mission are to be found in the following entire issues of *Journal of Geophysical Research*, 84:2795–3007 (June 1979), 84:7909–8519 (December 1979), and 82:3959–4680 (September 1977). For an excellent and comprehensive review of the geologic evolution of Mars, see RAYMOND E. ARVIDSON, KENNETH A. GOETTEL, and CHARLES M. HOHENBERG, "A Post-Viking View of Martian Geologic Evolution," in *Reviews of Geophysics and Space Physics*, 18:565–603 (August 1980). An overview of the morphology of surface structures found by Viking and Mariner 9 is MICHAEL H. CARR, "The Morphology of the Martian Surface," in *Space Science Reviews*, 25:231–284 (1980). A detailed account of factual knowledge about the Martian satellites is JOSEPH VEVERKA and JOSEPH A. BURNS, "The Moons of Mars," in *Annual Review of Earth and Planetary Sciences*, 8:527–558 (1980). Knowledge of the meteorology of the Martian atmosphere is described in CONWAY B. LEOVY, "Martian Meteorology," in *Annual Review of Astronomy and Astrophysics*, 17:387–413 (1979). ARTHUR KOESTLER, *The Sleepwalkers: A History of Man's Changing Vision of the Universe* (1959), gives a magnificent account of the role of Mars in the work of Brahe and Kepler. Maps of Mars can be obtained from the superintendent of the U.S. Geological Survey, Washington, D.C.

(M.J.S.B.)

Jupiter. A discussion of telescopic observations of Jupiter consisting of descriptions of motions, colours, and transformations observed in the Jovian cloud layers may be found in B.M. PEEK, *The Planet Jupiter* (1958). An extensive compilation of papers on all aspects of the Jovian system through 1975 is presented in T. GEHRELS (ed.), *Jupiter* (1976). The Pioneer missions are described in R.O. FIMMEL, W. SWINDELL, and E. BURGESS, *Pioneer Odyssey: Encounter with a Giant* (1974). An early summary of information about the satellites appears in J.A. BURNS (ed.), *Planetary Satellites* (1977). A later account is given in D. MORRISON (ed.), *Satellites of Jupiter* (1981), which includes a discussion of the Voyager results pertaining to the satellites. Summary articles of other Voyager discoveries may be found in the following journals: *Science*, 204:945–957, 960–1008 (June 1, 1979), and 206:925–996 (November 23, 1979); *Nature*, 280:725–806 (August 30, 1979); *J. Geophys. Res.*, 86 (September 30, 1981), the entire issue being devoted to "Voyager Missions to Jupiter." The Voyager spacecraft and their experiments are described in *Space Science Reviews*, vol. 21, no. 2 and 3 (1977). An overview of this mission is given in D. MORRISON and J. SAMZ, *Voyage to Jupiter* (1980); and many of the results and their interpretation are presented in articles in J.K. BEATTY, B. O'LEARY, and A. CHAIKIN (eds.), *The New Solar System* (1981).

(T.C.O.)

Saturn: A.F.O'D. ALEXANDER, *The Planet Saturn: A History of Observation, Theory and Discovery* (1962, reissued 1980), contains a list of all important papers published up to 1962. For an overview of information gathered by the Voyager space probes, see DAVID MORRISON, *Voyages to Saturn* (1982). See also *Science*, 212:159–243 (April 10, 1981); *Science*, 215:499–594 (January 29, 1982); and ANDREW P. INGERSOLL, "Jupiter and Saturn," *Sci. Am.*, 245:90–108 (December 1981).

(P.Mo./Ed.)

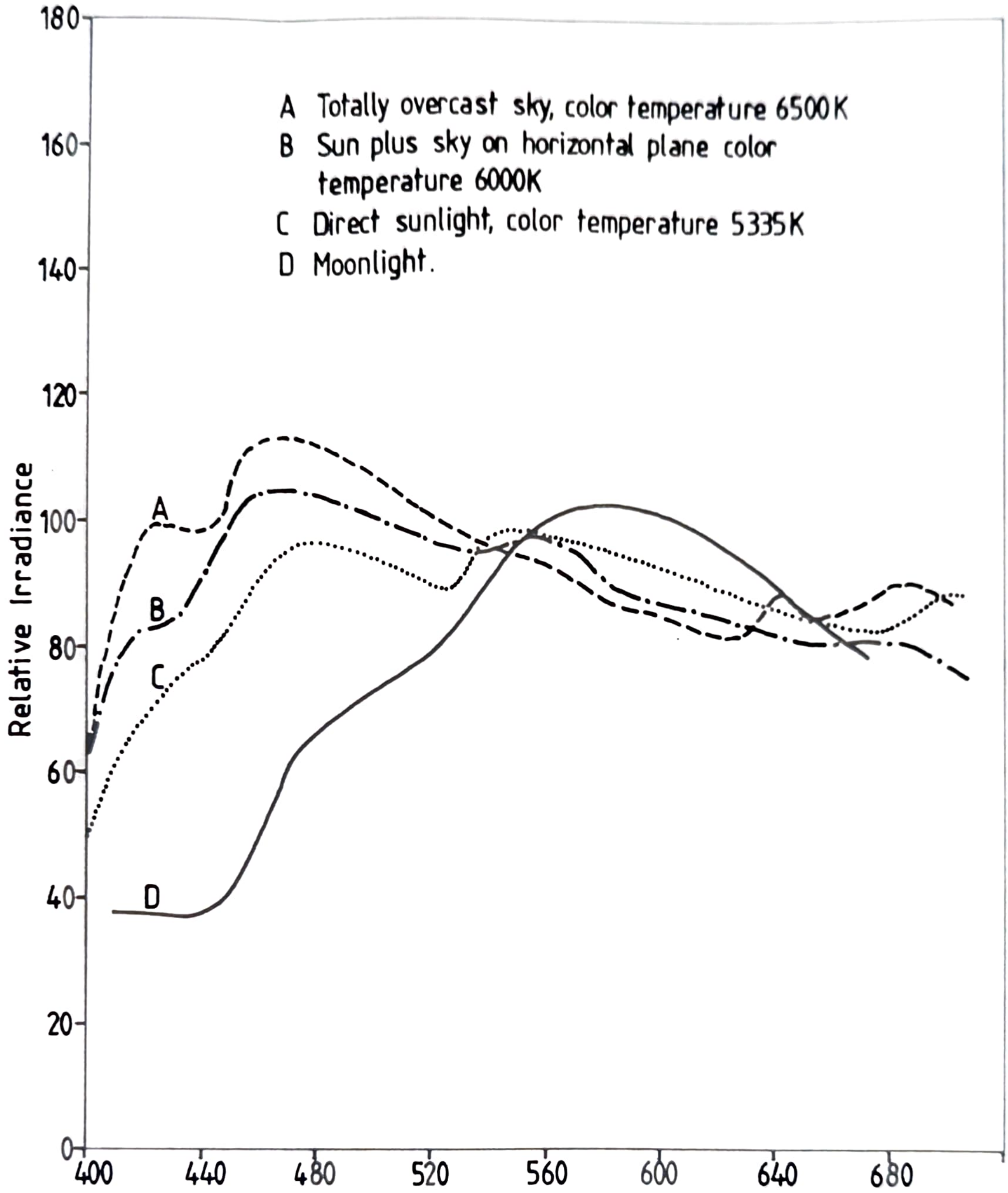
Uranus: A comprehensive and detailed summary of essentially all the serious work done on Uranus from its discovery through the early 1960s is ARTHUR F.O'D. ALEXANDER, *The*

Spectral Data

For wavelengths in visible spectrum and near infra-red (< 2000 nanometres) spectral content of moonlight approximately equal to spectral content of sunlight Moonlight very slightly yellower than sunlight.

For wavelengths in far infra-red (> 2000 nanometres) moon absorbs all radiation

Surface temperature of new moon	:	90 K
Surface temperature of full moon	:	390 K



Average spectral distribution curves for types of natural light on the basis of 'equal illumination'. (After Taylor and Kerr, and Vogel) given by Dandekar for normal zenith day sky, sky at mid-totality and morning twilight for $h = -4{\cdot}1°$ showed most difference below 550 nm and above 630 nm (thirteen wavelengths were recorded for each), and their corresponding colour temperatures reflected different scattering conditions when compared with the equivalence in luminance between eclipse and twilight conditions. The three curves gave 9100, 11 100 and 7800 K respectively. Spectral observations on this eclipse were made in South America by Velasquez,(437) and by Lloyd and Silverman who also reported measurements from an aircraft at 11·6 km over the south Atlantic.(438) Dandekar quoted earlier observations in the 15 500 to 16 400 K range, but found 11 000 K again during the eclipse of March 7, 1970.(439) Polarization values were consistent with the predominance of multiple scattering during the eclipse, as required by the luminance changes discussed by Lloyd and Silverman.

Shaw made polarization measurements at an eclipse in Kenya on June 30, 1973, when a fall of zenith luminance by a factor of 10^4 and a change to a more blue colour occurred. The horizon sky was seen to become more red.(440) Spectral changes in the zenith light during several eclipses were correlated by Hall.(441)

From the above it is clear that little effect on sky colour is likely during partial eclipses of moderate extent. Sastri observed the partial eclipse at Delhi on February 15, 1961, and measuring the sky chromaticity at 60° from the sun by spectroradiometry (Chapter 10), found no significant change during the eclipse compared with changes after the eclipse.(442) He quoted Jagannathan *et al.* who, unlike other authors, found the sky to become less blue during a partial eclipse (December 12, 1955). They used blue, yellow and red filters for measurement.(443)

THE MOON

The moon reflects a small amount of sunlight from its dark surface rocks, an amount which would be increased by fourteen times if the surface were a perfect diffuser. This albedo of 0·07 is the mean value for a surface showing marked directional reflection which increases the observed luminance of the full moon above that calculated from the data of Chapter 3, namely 3×10^3 cd/m². Under clear atmospheric conditions the luminance may reach 4×10^3 cd/m², or more than half that of an average 40 W fluorescent tube, sufficient for its use in

spectrographic determination of atmospheric ozone, and in the mapping of its own infra-red emission spectrum which began with Langley in 1884,(444) and continued with Adel and others.(445) The infra-red emission varies with the moon's phase since the surface temperature varies from about 90 to 390 K during the cycle.

As a source of light the interest of the moon lies in its effects on the extraterrestrial solar spectrum. In Chapter 7 some of the earliest spectrophotometry by Vogel was described, including his work on moonlight.(196) His measurements at eight wavelengths 426, 444, 464, 486, 517, 555, 600 and 633 nm may be expressed as ratios of moonlight to sunlight, normalized at 555 nm. Vogel also measured sunlight reflected from different rocks. The ratios derived from a yellow-grey sandstone are compared with the other ratios.

Moon:sun	0·56,	0·50,	0·62,	0·68,	0·84,	1·00,	1·07,	1·06
Sandstone:sun	—,	0·58,	0·75,	0·73,	0·87,	1·00,	1·09,	1·11

Thus moonlight was shown to be somewhat yellowish compared with sunlight, and its surface might well resemble the yellow-grey sandstone.

At the same period of time Pickering found a much more marked tilting of the solar spectrum after reflection from the moon.(39) It is possible that his sunlight was mixed with skylight though this would hardly account for the large decrease in the violet band. Pickering's wavelengths were roughly 450, 520, 590 and 650 nm. The relative values for sunlight and moonlight respectively were

2971:250:100:45 and 363:155:100:87.

Seventy years after these pioneers, Stair and Johnston made an accurate spectral survey using the same equipment as in the solar work in New Mexico (Chapter 9).(276) The intention was primarily to measure the ultra-violet radiation, and the long wavelength limit was 530 nm. No complete record of the absorptive effects of the lunar surface was obtained, but a selective absorption at 380 to 390 nm was established, suggesting a yellowish glassy surface material. The samples recovered by the Apollo 11 Mission in July 1969 have confirmed the earlier deductions from spectrophotometry. Many other investigations show the ratio between the spectra of sunlight and moonlight when normalized at 500 nm: there is a steadily increasing advantage to moonlight in the wavelength range of 500 nm to 2·4 μm, and to sunlight from 500 nm to 300 nm, hence the slightly more yellow colour of moonlight.[446]

When the moon is just past the new phase and is visible soon after sunset, most of its disc is illuminated by light from the earth, the bluish white light mentioned in Chapter 4. Owing to the earth's higher albedo it is much more bright, seen from the moon, than is the moon seen from the earth, and this reflected earthlight illuminates the dark part of the moon to a noticeable extent. It is called the 'ash-grey' or 'ashen' light. This term is also applied to the darker part of the planet Venus when it presents an illuminated crescent, though the cause is disputed in this case. For the moon the ashen light is not the same as the dull reddish appearance in total eclipse, when sunlight is refracted through the earth's atmosphere on to the disc but with loss of the shorter wavelengths by scattering in the long air path. This was explained by Kepler in about 1600, and the ashen light by Leonardo da Vinci a century before.

In contrast to these early discerning views there are others where primitive ideas persisted among reputable astronomers of much later date. For example, though it was not unreasonable for the eighteenth century William Herschel to believe the moon to be inhabited, it is astonishing to find the astronomer Pickering asserting, in 1924, that apparent changes in the lunar crater Eratosthenes could be explained by temporary snow fields and the movements of large groups of small animals.[447]

THE ZODIACAL LIGHT AND OTHER SUNLIGHT EFFECTS

A minute proportion of the sun's light reaches the earth in the zodiacal light, always present, but not well known because of the difficulty of seeing its feeble luminance which is of the order of 10^{-2} cd/m^2.[448] No appreciable illumination of the earth can be seen. On winter evenings in the western sky, and in autumn before dawn in the east, this faint pyramid of light extends along the plane of the ecliptic. It is probably sunlight scattered by dust or electrons in this plane. The Gegenschein is an even fainter glow of similar origin. Minnaert gives an attractive account of these and other atmospheric phenomena.[449] More colourful ones are due to atmospheric refraction and aerosol scattering, for

example the green flash from the setting sun,[450, 451] and the many appearances of blue or green sun and moon. One record is due to Edward Whymper who, when climbing Chimborazo in Ecuador, in 1880, saw an eruption of Cotopaxi, 60 miles away. The clouds of volcanic dust drifted between the sun and the mountaineering party, causing the sun to appear green, with other brilliant colour effects in the sky.[452] Soon afterwards the volcanic explosion of Krakatoa in the Sunda strait produced a few years of spectacular sunrises and sunsets by the vast amounts of aerosols released by the explosion, and appearances of blue or green sun and moon were frequent.[453] The 1963 eruption of Mount Agung in Bali had similar results.

Volcanic dust is considered to be the most effective of the aerosols causing loss of solar radiation to the earth, with the result that the sea level temperature is lowered by an estimated 1·3 K.[454] The quantity of naturally produced aerosol in the atmosphere has been put at $2{\cdot}3 \times 10^{12}$ kg per annum, or about eight times that from man-made sources.[455]

The spectral distribution of the zodiacal light[456] and the Gegenschein[457, 458] have been examined to a limited extent. For the green flash, Jacobsen measured diffraction grating spectra on colour cine film, taken at sunset over the Pacific ocean at Oahu (Hawaii). The resulting curves of visual intensity showed a maximum near 525 nm for the flash, whereas the low sun gave a flatter curve with its maximum near 505 nm, confirming the excess of green light in the flash.[459] Soon afterwards, in Edinburgh, Wilson photographed the spectrum of a 'deep indigo blue' sun and attributed the lack of red wavelengths to a layer of fine particles lying between 9·5 and 13 km above the earth. The particles were probably oil globules from a forest fire three days earlier in Alberta, Canada. The cloud remained long enough to produce a blue moon the same night.[460]

Sunrise, twilight and sunset offer a great variety of colour effects which have scarcely been considered except qualitively.[449] In Chapter 10 some spectroradiometry of the horizon sky at these times was described, but there seems to be no systematic account of the range and variations of chromaticity found in the sky when scattering effects are most prominent, possibly because of the apparently infinite variety to be seen.

To return to the night sky: other light production comes from fluorescence of excited atoms or recombination of ions liberated in

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C. W. Allen Section OY30

CHAPTER 9

SUN

§ 75. Sun Dimensions

Sun radius	$\mathscr{R}_{\odot}$	$= 6.9599(7) \times 10^{10}$ cm
Volume	$V_{\odot}$	$= 1.4122 \times 10^{33}$ cm^3
Surface area		$= 6.087 \times 10^{22}$ cm^2
Sun mass	$\mathscr{M}_{\odot}$	$= 1.989(2) \times 10^{33}$ g
Mean density	$\bar{\rho}_{\odot}$	$= 1.409$ g cm^{-3}
Gravity at surface		$= 2.7398(4) \times 10^4$ cm s^{-2}
Centrifugal acceleration at equator		$= -0.587$ cm s^{-2}
Radiation emitted	$\mathscr{L}_{\odot}$	$= 3.826(8) \times 10^{33}$ erg s^{-1}
Radiation emitted at surface	$\mathscr{F}$	$= 6.27 \times 10^{10}$ erg cm^{-2} s^{-1}
Moment of inertia [§ 76]		$= 5.7 \times 10^{53}$ g cm^2

Angular rotation velocity (at lat = 16°)
$= 2.865 \times 10^{-6}$ rad s^{-1}

Angular momentum (based on surface rotation)
$= 1.63 \times 10^{48}$ g cm^2 s^{-1}

Rotational energy (based on surface rotation)
$= 2.4 \times 10^{42}$ erg

Work required to dissipate solar matter to infinity [§ 76]
$= 6.6 \times 10^{48}$ erg

Sun's total internal radiant energy [1]
$= 2.8 \times 10^{47}$ erg

Translational energy (atoms and electrons) [1]
$= 2.7 \times 10^{48}$ erg

Escape velocity at Sun's surface $= 617.7$ km/s

General magnetic field near Sun's pole (at spot minimum)
$\simeq 1$ or 2 gauss

Magnetic flux from polar area at spot min.
$\simeq 8 \times 10^{21}$ maxwell

Viewed from the Earth

Mean equatorial horizontal parallax [§ 10]
$= 8''.79418$
$= 4.26354 \times 10^{-5}$ rad

Mean distance from Earth (= astronomical unit, § 10)
AU $= A = 1.495979(1) \times 10^{13}$ cm
$= 92.9558 \times 10^6$ miles

Distance at

perihelion	$= 1.4710 \times 10^{13}$ cm
aphelion	$= 1.5210 \times 10^{13}$ cm

Semi-diameter of Sun at mean Earth distance

$= 959''.63$

$= 0.0046524$ rad

Semi-diameter plus irradiation (for observing limb)

$= 961''.2$

Oblateness. Semi-diameter difference (equator–pole) [2]

$= 0''.05$

Solid angle of Sun at mean distance $= 6.8000 \times 10^{-5}$ sr

$A/\mathscr{R}_\odot = 214.94$

$(A/\mathscr{R}_\odot)^2 = 46200$ $\quad (A/\mathscr{R}_\odot)^{1/2} = 14.661$

Surface area of sphere of unit radius

$4\pi A^2 = 2.8123 \times 10^{27}$ cm^{-2}

On solar hemisphere $1° = 12147$ km

At mean distance A

$1'$ of arc $= 4.352 \times 10^4$ km

$1''$ of arc $= 725.3$ km

Sun as a star

Magnitude [1, 4, 7, 8, 9]

	Apparent	Modulus	Absolute
Visual	$m_v = V = -26.74$	31.57	$M_v = +4.83$
Blue	$B = -26.09$		$M_B = +5.48$
Ultraviolet	$U = -25.96$		$M_U = +5.61$
Bolometric	$m_{bol} = -26.82$		$M_{bol} = +4.75$

Colour indices [1, 3, 4, 5, 6, 7, 9]

$B-V = +0.65$

$U-B = +0.13$

$U-V = +0.78$

$V-R = +0.52$

$V-I = +0.81$

$V-K = +1.42$

$V-M = +1.53$

Bolometric correction BC $= -0.08$

Spectral type $= G2\ V$

Effective temperature $= 5770$ °K

Velocity relative to near stars $= 19.7$ km/s

Solar apex $A = 271°$ $\quad D = 30°$ (1900)

$L^{\text{II}} = 57°$ $\quad B^{\text{II}} = 22°$

Age of Sun $= 5 \times 10^9$ y

i.e. a little greater than age of Earth.

[11] R. T. Parnell and J. M. Beckers, *Sol. Phys.*, **9**, 35, 1969.
[12] D. Labs and H. Neckel, *Z. Ap.*, **69**, 1, 1968.
[13] E. Gurtovenko and V. Troyan, *Sol. Phys.*, **20**, 264, 1971.
[14] O. O. Badalyov and M. A. Livshits, *Sol. Phys.*, **22**, 297, 1972.
[15] C. de Jager and L. Neven, *Sol. Phys.*, **22**, 49, 1972.

§ 79. The Strong Fraunhofer Lines

W = equivalent width, r_c = central or minimum intensity corrected for instrumental distortion, c = wing intensity defined by $c = \Delta\lambda^2(1-r)/r$ [1, 9] where r is the intensity (not the depth) relative to the continuum at $\Delta\lambda$ from the line centre. The limb is represented by $\cos\theta = 0.3$ where θ is angular distance from disk centre. Between $\cos\theta$ = 0.3 to 0.0 most features change rapidly.

λ	Name	Atom	Centre of disk			Limb ($\cos\theta = 0.3$)		Ref.
			W	r_c	c	W	r_c	
Å			Å	%	Å^2	Å	%	
2795.4		Mg II	22 (2795.4 and 2802.3)	10				
2802.3		Mg II		10				
2851.6		Mg I	10	10				[2]
2881.1		Si I	2.6	20				
3581.209	N	Fe I	2.2	3				
3734.874	M	Fe I	3.1	1				
3820.436	L	Fe I	1.8	2				
3933.682	K	Ca II	19.2	3.9	39	16	8	[4, 10]
3968.492	H	Ca II	14.4	4.1	26	12	8	[4]
4045.825		Fe I	1.2	2	0.22	1.4	5	
4101.748	h, Hδ	H I	3.4	19		1.2	31	[3, 4]
4226.740	g	Ca I	1.5	2.4	0.23	1.5	4	[4]
4340.475	G′, Hγ	H I	3.5	17		1.2	26	[3]
4383.557	d	Fe I	1.1	3		1.1	5	
4861.342	F, Hβ	H I	4.2	14		1.4	22	[3, 4]
5167.327	b_4	Mg I	0.9	12	0.09	0.7	18	
5172.698	b_2	Mg I	1.3	8	0.24	1.2	11	[4]
5183.619	b_1	Mg I	1.6	7	0.37	1.5	11	
5889.973	D_2	Na I	0.77	4.2	0.095	0.76	6	[4]
5895.940	D_1	Na I	0.57	4.8	0.049	0.56	6	
6562.808	C, Hα	H I	4.1	16		1.4	23	[3, 4]
8498.062		Ca II	1.3	30	0.3	1.1	32	[7]
8542.144		Ca II	3.6	19	2.4	2.9	20	
8662.170		Ca II	2.7	21	1.2	2.2	22	
10049.27	Pδ	H I	1.6	79				
10938.10	Pγ	H I	2.2	73		1.0	82	
12818.23	Pβ	H I	4.2	63				

[1] *A.Q.* **1**, § 68; **2** § 79.
[2] H. C. McAllister, *Atlas of Solar Ultraviolet 1800–2965 Å*, Upper Air Lab., Boulder, 1960.
[3] D. M. Kuli-Zade, *Sov. A.*, **8**, 736, 1965; **9**, 788, 1966.
[4] H. Holweger, *Z. Ap.*, **65**, 365, 1967.
[5] O. R. White and Z. Suemoto, *Sol. Phys.*, **3**, 523, 1968.
[6] J. W. Brault *et al.*, *Sol. Phys.*, **18**, 366, 1971.
[7] C. de Jager and L. Neven, *B.A.N. Supp.*, **1**, 325, 1967.
[8] C. E. Moore *et al.*, *The Solar Spectrum 2935 to 8770 Å*, U.S. N.B.S. Mon. 61, 1966.
[9] E. A. Gussmann, *Z. Ap.*, **59**, 66, 1964.
[10] J. M. Pasachoff, *Sol. Phys.*, **19**, 323, 1971.

§ 80. Total Solar Radiation

Solar constant = flux of total radiation received outside Earth's atmosphere per unit area at mean Sun–Earth distance [1, 2, 3, 4]

$$f = 1.950(4)\ \mathrm{cal\ cm^{-2}\ min^{-1}}\ \text{(or langley/min)}$$
$$= 1.360 \times 10^{6}\ \mathrm{erg\ cm^{-2}\ s^{-1}}$$

Radiation from the whole Sun

$$\mathscr{L}_{\odot} = 3.826(8) \times 10^{33}\ \mathrm{erg\ s^{-1}}$$

Radiation per unit mass, $\mathscr{L}_{\odot}/\mathscr{M}_{\odot}$

$$= 1.924\ \mathrm{erg\ s^{-1}\ g^{-1}}$$

Radiation emittance at Sun's surface

$$\mathscr{F} = 6.284 \times 10^{10}\ \mathrm{erg\ cm^{-2}\ s^{-1}}$$

Mean radiation intensity of Sun's disk

$$F = \mathscr{F}/\pi = 2.000 \times 10^{10}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ sr^{-1}}$$

Radiation intensity at centre of disk

$$I_0 = 2.41 \times 10^{10}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ sr^{-1}}$$

Sun's effective temperature [5]

$$T_e = (\mathscr{F}/\sigma)^{1/4} = 5770\ °\mathrm{K}$$

Central disk temperature $(\pi I(0)/\sigma)^{1/4}$ [5]

$$= 6050\ °\mathrm{K}$$

Mean brightness of Sun's disk outside atmosphere

$$= 1.98 \times 10^{5}\ \text{stilb}$$

Brightness of centre of disk outside atmosphere

$$= 2.48 \times 10^{5}\ \text{stilb}$$

Candle power of the Sun

$$= 2.84 \times 10^{27}\ \text{cd}$$

Light flux outside atmosphere at mean solar distance

$$= 12.7\ \text{phot} = 127000\ \text{lux}$$

[1] *A.Q.* **1**, § 69; **2**, § 80.
[2] D. Labs and H. Neckel, *Sol. Phys.*, **19**, 3, 1971.
[3] D. Labs and H. Neckel, *Z. Ap.*, **69**, 1, 1968.
[4] A. J. Drummond *et al.*, *Nature*, 218, 259, 1968; *Science*, **161**, 888, 1968.
[5] D. Labs and H. Neckel, Private communication, 1972.

§ 81. Solar Limb Darkening

$I'_\lambda(\theta)$ = intensity of solar continuum at angle θ from the centre of the disk; θ = angle between Sun's radius vector and the line of sight.

$I'_\lambda(0)$ = continuum intensity at centre of disk.

The ratio $I'_\lambda(\theta)/I'_\lambda(0)$, which varies with the wavelength λ, defines *limb darkening*. As far as possible measurements are made in the continuum between the lines (hence primes ′ in the notation).

where favourable climatic conditions were to be expected.[274, 275] In the first phase of the work the combination of a quartz double monochromator, a photocell with peak sensitivity at about 375 nm (RCA 935), a light modulator, amplifier and recorder had proved adequate to record fine structure of ultra-violet Fraunhofer lines in the solar spectrum. A siderostat reflected the light on to the entrance slit without other optical components. Calibration of the energy scale was effected by a tungsten-in-fused-silica lamp of known colour temperature, but using spectral emissivity data for tungsten in the region 300 to 430 nm. This range was extended to 535 nm at the mountain station. Stair discussed the possible polarization error due to reflections in the siderostat: there was no significant dependence of the effect on wavelength but intensity changes of ± 5 per cent were observed between morning and afternoon measurements at $m=3$.

The rapid spectral scan in 2 to 3 minutes, with eight successive amplifier sensitivity ranges for different wavelength ranges, dispensed with the need for monitoring the total light flux or some fixed wavelength band to provide a correction for fluctuating irradiation. Repeated observations were made daily during the three week operation in September 1951, and from these the material from four days was selected for the calculations. Atmospheric transmittance was determined by Bouguer lines at twenty-five selected wavelengths (Fig. 14), and irradiance values extrapolated to $m=0$. The tabulated results also included values for $m=1, 2$ and 3.

With a different double monochromator and an RCA 1P28 photomultiplier, Stair and Johnston next measured ultra-violet radiation from the moon when near full.[276] Comparison with the Colorado observations revealed a general absorption by the declining reflectance from 400 to 320 nm. This was interrupted by a minimum due to a selective absorption at 380 to 390 nm, a possible indication of the surface material (see Chapter 15). Atmospheric transmittance was calculated again, to give a straight line relation between the logarithms of wavelength and transmittance between 325 and 550 nm.

The next use of this equipment, with the original photocell, took place at Sacramento Peak, near Sunspot, New Mexico (2800 m). Spectral irradiance measurements were within 5 per cent of those found at Chalk Mountain, and the ozone content was the same 2·1 mm. A lead sulphide photoconductive detector extended the range to 2·5 μm.

suspended particles as large as, or larger than, the wavelengths of visible light, where Rayleigh's theory does not apply. In some of the early work it appears that the spectral distribution curve for the blue sky was expected to show an inverse fourth power distribution, though this must depend on the spectrum of the incident sunlight which was not well known at the time. The reports of these investigators are worth reading, but are sometimes hardly worth the effort of trying to unravel their experiments in the hope of discovering useful information in modern terms.

The blue colour of the sky, considered more fully in Chapter 15, is the most obvious result of Rayleigh scattering. Because the scattering occurs in all directions, some of the incident sunlight is redirected outwards from the atmosphere, forming one component of the earth's albedo. It produces the bluish halo seen from outside the atmosphere, making earthlight on average more blue than moonlight where no such scattering occurs.

The quantitative aspect of scattering is conveniently expressed as an extinction or attenuation of incident light according to the principle first established by Bouguer.(109) He expressed it in this form: when light traverses different thicknesses of the same body, the difference of the logarithms of the quantities of light (incident and emergent) is always in the same ratio to the thickness of the medium traversed. Both types of logarithm are used in the mathematical statement, but usually it is expressed as:

$$I/I_0 = \exp(-al)$$

for a thickness l and an absorption coefficient a. This law was also discovered by Lambert, expressed by him more mathematically, and is often given his name. Bouguer's discovery was the earlier, and his complete work was published posthumously in the same year as Lambert's statement of the law (1760).

The coefficient a usually varies with wavelength, and for Rayleigh scattering is proportional to $1/\lambda^4$. Necessary modifications to the theory appear later, but its use in a simple form has been most important in calculations of the solar constant (Chapter 5). For this the logarithm of H_λ, the relative spectral power distribution of sunlight, is plotted versus the air mass m, or sec z where z is the sun's zenith angle. Each wavelength gives a nearly straight line, the more

58. Find the circle which is orthogonal to all the circles
$x^2 + y^2 + 2x + 4y + 3 = 0$, $x^2 + y^2 + 3x + 5y = 0$, $x^2 + y^2 + 4x + 5y - 1 = 0$.

59. Find the two circles each of which cuts $x^2 + y^2 - 2x + 2y - 7 = 0$ and $x^2 + y^2 = 3$ at right angles and touches the line $y = 1$.

60. Find the limiting points of the system given by

$$x^2 + y^2 - 2\lambda(x + y - 4) - 6 = 0.$$

Determine the equations of the circles which pass through these points and have radius 2 units.

61. A triangle has vertices O (0, 0), A (4, 2), B (4, 6). Verify that the circle which passes through the mid-points of the sides has the equation

$$x^2 + y^2 - \tfrac{15}{2}x - 4y + 14 = 0.$$

Prove that this circle also passes through L and M, the feet of the perpendiculars from O and B on to the opposite sides of the triangle.

The altitudes OL and BM meet in H. Prove that the coordinates of H are (7, 0) and find the coordinates of the points at which the line AH meets the circle. (O.C.)

62. Find for what values of k the circles

$$x^2 + y^2 + 4x - 2y - 5 = 0, \quad x^2 + y^2 + x - 3y + k = 0$$

touch, and determine in each case whether the circles touch internally or externally. (O.C.)

CHAPTER V

THE PARABOLA

38. The parabola

A parabola may be defined as the locus of a point P which moves in the plane of a point S and a straight line d so that $SP = PM$, where PM is perpendicular to d (Fig. 17). The point S is called the *focus* and the line d the *directrix* of the curve.

Draw SX through S perpendicular to d and through P draw PN perpendicular to SX and produce it to P' so that $PN = NP'$. If $P'M'$ is the perpendicular from P' to d, the figure $MPP'M'$ is a rectangle and $P'M' = PM$. Further, SN is the perpendicular bisector of PP' and

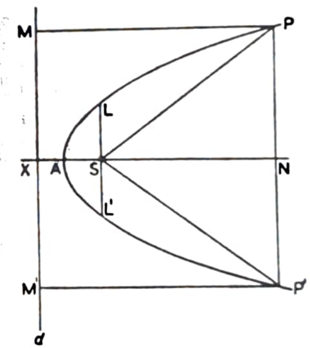

Fig. 17.

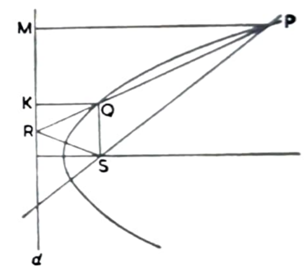

Fig. 18.

therefore $SP' = SP = PM = P'M'$. It follows that P', which is the image of P in XS, is on the parabola and that the curve is symmetrical about XS. The line XS is called the *axis* of the parabola.

The mid-point A of XS is equidistant from the focus and the directrix and is therefore on the parabola; it is called the *vertex*.

The chord LSL' through S which is parallel to d is the *latus rectum*; we shall denote its length by $4a$. From the definition of the curve it follows that $SL = SX = 2a$.

If Q is any other point on the curve, $SQ = QK$, where QK is at right angles to d (Fig. 18). Let PQ meet d at R, then

$$SP/SQ = PM/QK = PR/QR,$$

because the triangles PRM, QRK are similar. Now consider the triangle SPQ; its side PQ is produced to R so that R divides PQ externally in the ratio of the sides SP and SQ. It follows that SR is the exterior bisector of the angle PSQ.

If now P is kept fixed and Q is moved along the parabola so as to approach coincidence with P, the chord PQ approaches a limiting position defined to be the *tangent* at P to the parabola. When Q coincides with P the angle PSQ becomes zero and its exterior bisector (the limiting position of SR) is at right angles to the coincident lines SP and SQ. Thus *the part of the tangent between the curve and the directrix subtends a right angle at the focus.*

Let the tangent at P cut d at Z and the axis at T, and let SM cut PT at Y (Fig. 19). Then, in the right-angled triangles SZP, MZP we have PZ as common hypotenuse and the sides SP, PM equal (from the definition of the parabola). Hence the triangles are congruent and $\angle SPT = \angle MPT$, and it follows that the tangent at P bisects the angle SPM. This gives a simple method of constructing the tangent at a given point on the curve (the curve itself is regarded as determined when the focus and directrix are known).

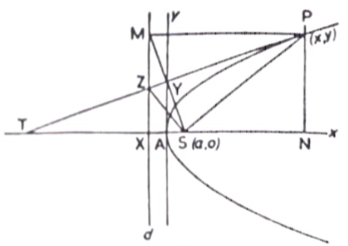

Fig. 19.

In particular, consider the tangent at the vertex A. If P is moved along the curve until it coincides with A, the angle SPM becomes the angle SAX which is equal to two right angles. It follows that the tangent at A is the line Ay which is perpendicular to the axis (and so parallel to the directrix).

Since PM is parallel to the axis, $\angle STP = \angle MPT = \angle SPT$. The triangle STP is therefore isosceles and $SP = ST$. From this result we deduce another method of constructing the tangent at any point of the curve.

A being the mid-point of SX and AY being parallel to XM, it follows that Y is the mid-point of SM which is the base of the isosceles triangle SPM; so SYP is a right angle, or, in other words, SY is the perpendicular from S on to the tangent at P. It follows that the locus of the foot of the perpendicular drawn from the focus to a variable tangent to the parabola is the tangent at the vertex.

The triangle TAY being right-angled at A, the circle on TY as diameter passes through A and SY is its tangent at Y. Therefore

$$SY^2 = SA.ST = SA.SP.$$

39. The equation of a parabola

Let N be the foot of the perpendicular from P to the axis (Fig. 19). Then $XNPM$ is a rectangle and $PM = XN = XA + AN = AS + AN$.

Also $SP^2 = SN^2 + PN^2 = (AN - AS)^2 + PN^2 = PM^2 = (AN + AS)^2$.

Hence $\quad PN^2 = (AN + AS)^2 - (AN - AS)^2 = 4AS.AN \quad \ldots\ldots (1)$

Take the vertex A as origin, the axis of the parabola as x-axis, and the tangent at A as y-axis. If P is the point (x, y) on the curve, so that $AN = x$ and $PN = y$, the above relation becomes

$$y^2 = 4ax, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where $a = AS$ and the latus rectum is of length $4a$. This is the equation of the curve referred to the axis and the tangent at the vertex.

The focus is the point $(a, 0)$ and the equation of the directrix is

$$x = -a.$$

40. Parametric coordinates

Putting $y = 2at$ in equation (2), we find that $x = at^2$; it follows that, for all values of t, the point $(at^2, 2at)$ lies on the curve; t is the *parameter* of the point. If t increases steadily from $-\infty$ to $+\infty$, the point describes the curve from the negative to the positive side of the axis.

The parameter of the vertex $(0, 0)$ is zero; the parameters of the ends of the latus rectum are ± 1.

The advantage of using parametric coordinates for a point on the curve is that the point is completely specified as a point on the curve by its parameter. The point $(at^2, 2at)$ must necessarily lie on the parabola. We could of course refer to it as the point (x', y'), where $y'^2 = 4ax'$. But this would obviously be cumbrous.

41. Chords

The chord which joins the points whose parameters are t and t', *i.e.* the points $(at^2, 2at)$ and $(at'^2, 2at')$ is given by

$$\frac{x - at^2}{a\,(t^2 - t'^2)} = \frac{y - 2at}{2a\,(t - t')},$$

which, after removal of the common factor $a\,(t - t')$ in the denominators, gives

$$\frac{x - at^2}{t + t'} = \frac{y - 2at}{2}.$$

Multiplying by $t + t'$ and rearranging, we get

$$x - at^2 = \tfrac{1}{2}(t + t')\,y - at^2 - att'$$

and finally $\quad x - \tfrac{1}{2}(t + t')\,y + att' = 0, \quad \ldots\ldots\ldots\ldots\ldots (3)$

as the equation of the chord whose ends are the points t and t' (it is often convenient to refer to a point on the curve by stating its parameter).

42. Tangents

If t is fixed and t' is made to approach the value t, the point t' moves along the curve towards the fixed point t and the chord (3) approaches the position given by the equation

$$x - ty + at^2 = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

which is obtained by putting $t' = t$ in (3). This gives the tangent whose point of contact is the point t.

If $t \neq 0$, the equation may be written

$$y = \frac{x}{t} + at, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

from which it follows that the slope of the tangent at t is $1/t$. (If $t = 0$ the tangent is $x = 0$, the y-axis.)

43. Condition of tangency

Suppose that the line

$$lx + my + n = 0$$

is a tangent to the parabola. Then, if the point of contact is t, this line is identical with that given by (4). The coefficients in the two equations are therefore proportional and

$$\frac{1}{l} = \frac{-t}{m} = \frac{at^2}{n},$$

whence $$t = -m/l = -n/am$$

and $$am^2 = ln \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

is the condition of tangency. If the condition is satisfied, the parameter of the point of contact is $-m/l$.

Examples

1. The tangent at the point L $(a, 2a)$ is found from (4) by putting $t = 1$; it is $x - y + a = 0$, or $y = x + a$. It makes 45° with the x-axis.

2. From equation (4) it is seen that the tangent at P $(at^2, 2at)$ cuts the x-axis at T $(-at^2, 0)$. Hence $ST = a + at^2$. The perpendicular PM from P to the directrix $(x + a = 0)$ is of length $a + at^2$. So $SP = ST = PM$.

3. Find the equation and point of contact of the tangent to the parabola $y^2 = 8x$ which is parallel to $y = -3x$.

For this parabola, $a = 2$ and the tangent at $(2t^2, 4t)$ is

$$y = \frac{x}{t} + 2t.$$

The slope of the required tangent being -3, it follows that $t = -\frac{1}{3}$ and the equation is

$$y = -3x - \tfrac{2}{3}.$$

The point of contact is $(\frac{2}{9}, -\frac{4}{3})$.

4. The perpendicular from the focus S meets the tangent at P in Y. Prove that $SY^2 = a.SP$ and that Y lies on the tangent at the vertex.

Let P be the point t on the parabola; then the tangent at P is

$$y = \frac{x}{t} + at$$

and SY, being the perpendicular to this through S $(a, 0)$, is

$$y = -tx + at.$$

Subtracting these two equations we obtain the equation of a line which passes through the point of intersection Y (see Section 18); it is

$$0 = \left(t + \frac{1}{t}\right)x.$$

The bracketed factor not being zero, this gives $x = 0$ which represents the tangent at the vertex.

The y-coordinate of Y is at; so Y is $(0, at)$ and

$$SY^2 = a^2(1 + t^2) = a.SP,$$

since $SP = PM = a + at^2$ (see Example 2 above).

5. Find the point in which the connector of the point t on the parabola to the focus S meets the curve again. Show that the tangents at these two points meet at right angles on the directrix.

Let the line meet the parabola again in the point t'. Then its equation is (3), Section 41. Since this line passes through S $(a, 0)$,

$$a + att' = 0$$

and therefore $t' = -1/t$.

The tangents at t and t' are respectively

$$y = x/t + at, \quad y = -tx - a/t.$$

These two lines are at right angles; for the product of their slopes is -1. On subtracting the two equations we have, as the equation of a line through their intersection (see Section 18),

$$x\left(t + \frac{1}{t}\right) + a\left(t + \frac{1}{t}\right) = 0.$$

On removal of the factor $t + 1/t$, which does not vanish for any real t, this becomes $x + a = 0$, and this is the equation of the directrix. So the two tangents at the ends of a focal chord meet at right angles on the directrix.

6. Find the equation of the chord of which the mid-point is (h, k).

Let the ends of the chord be the points t and t' on the parabola; then the slope of the chord $2/(t + t')$. The y-coordinate of the mid-point is $k = a(t + t')$; so the slope is $2a/k$. Since the chord passes through (h, k) it is given by

$$y - k = (2a/k)(x - h),$$

or

$$2ax - ky - 2ah + k^2 = 0.$$

In particular, if the given point (h, k) is on the curve, the chord is of zero length and is therefore the tangent at (h, k). In this case, $k^2 = 4ah$ and the equation becomes

$$2ax - ky + 2ah = 0.$$

7. Find the locus of the mid-points of all chords which have slope $\tan\theta$.

From equation (3) it is seen that the slope of the chord tt' is $2/(t + t')$. The y-coordinate of the mid-point of this chord is $a(t + t')$. So if the chord is of slope $\tan\theta$, its mid-point lies on the line

$$y = 2a \cot\theta,$$

which is parallel to the axis of the parabola. It meets the curve in the point whose parameter is $\cot\theta$.

Thus the mid-points of a system of parallel chords lie on a line parallel to the axis of the parabola. Such a line is said to be a *diameter* of the curve.

8. Find the locus of the mid-point of a variable focal chord.

In Example 6 the chord which has (h, k) as its mid-point was found to be

$$2ax - ky - 2ah + k^2 = 0.$$

If this is a focal chord it passes through the focus $(a, 0)$ and therefore

$$2a^2 - 2ah + k^2 = 0.$$

It follows that, if the chord moves so as always to pass through the focus, its mid-point always lies on the curve

$$2a^2 - 2ax + y^2 = 0.$$

Writing this equation in the form

$$y^2 = 2a(x - a)$$

we see that the locus is a parabola of latus rectum $2a$ and vertex at $(a, 0)$, its axis coinciding with that of the original parabola.

44. Normals

The line through a point P on the parabola and perpendicular to the tangent at P is defined to be the *normal* at P. If t is the parameter of P, the slope of the tangent at P is $1/t$; so that of the normal is $-t$. The equation of the normal is therefore

$$y - 2at = -t(x - at^2),$$

or

$$tx + y - 2at - at^3 = 0 \quad \dots\dots\dots\dots\dots (7)$$

Examples

9. The normal at $L(a, 2a)$ is found from (7) by taking $t = 1$. It is

$$x + y - 3a = 0.$$

10. The normal at $P(at^2, 2at)$ cuts the axis at $G(2a + at^2, 0)$ as is seen from (7) by putting $y = 0$. Hence $SG = a + at^2 = ST = SP = PM$ (from Example 2).

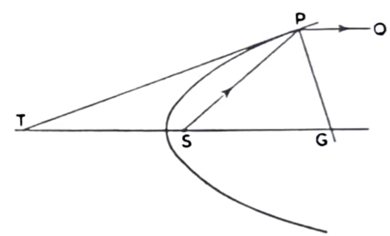

Fig. 20.

Further, if N is the foot of the perpendicular from P to the axis, so that N is $(at^2, 0)$, $NG = 2a$. Thus the orthogonal projection of PG on the axis is of constant length.

Since $SG = SP$, the angles SPG, SGP are equal. If then PQ (Fig. 20) is parallel to the axis, the angles SPG, GPQ are equal. This gives an important optical property of the parabola: if the curve reflects light in its own plane according to the optical law that the angles of incidence and reflection are equal, a ray from S to P is reflected along PQ. So if a point source of light is placed at the focus S, all rays from it are reflected in a beam parallel to the axis. The same property is possessed by the surface generated by rotating the parabola about its axis (the paraboloid of revolution); consequently paraboloidal mirrors are used for searchlights and headlights which are required to give a parallel beam. Conversely, a beam of rays of heat parallel to the axis will be reflected in rays which all pass through S where the heating effect will be concentrated. This accounts for the name "focus" or "fireplace".

11. Find the point in which the normal at t meets the curve again.

Let the normal meet the curve again at t'. Then the chord tt' is given by

$$x - \tfrac{1}{2}(t + t')\,y + att' = 0.$$

Being the normal at t, this line must be at right angles to the tangent

$$x - ty + at^2 = 0$$

and therefore

$$1 + \tfrac{1}{2}t\,(t + t') = 0,$$

whence

$$t' = -t - \frac{2}{t}.$$

12. Find the locus of the mid-point of a normal chord of the parabola.

A normal chord is one which is normal to the parabola at one of the points in which it meets the curve; so we may take such a chord to be tt' in Example 11. The coordinates of the mid-point of this chord are given by

$$2x = a\,(t^2 + t'^2) = a\left[t^2 + \left(t + \frac{2}{t}\right)^2\right],$$

$$y = a\,(t + t') = -2a/t.$$

To obtain the equation of the locus of this point when t varies we eliminate t between these two relations. The second gives $t = -2a/y$; substituting this value for y in the first, we have

$$2x = a\left[\frac{4a^2}{y^2} + \left(\frac{2a}{y} + \frac{y}{a}\right)^2\right].$$

On multiplication by ay^2 and reduction, this becomes

$$2axy^2 = y^4 + 4a^2y^2 + 8a^4.$$

This relation is satisfied by the mid-points of all normal chords and therefore gives the required locus.

Exercises V

1. The line joining $P\,(at^2, 2at)$ to the focus S of the parabola $y^2 = 4ax$ meets the curve again at Q. Show that the parameter of Q is $-1/t$.

If A is the vertex and PA produced meets the line through Q parallel to the axis of the curve in M, prove that M is on the directrix and that SM is parallel to the tangent at P.

2. P is any point on a parabola. The tangent and normal at P meet the axis in T and G. Prove that the locus of the centroid of the triangle PGT is a parabola with latus rectum a third of that of the original parabola.

3. X and Z are the points of contact of two perpendicular tangents WX, WZ to a parabola and $WXYZ$ is a rectangle. If WX makes an angle $\tan^{-1} 2$ with the axis of the parabola, find the coordinates of Y.

4. The tangents at points P, Q on a parabola intersect at T and the line through T perpendicular to PQ meets the axis in M. Prove that the projection of TM on the axis is of constant length.

5. Q and R are the points of contact of two tangents PQ, PR to the parabola $y^2 = 4ax$. If the ratio of the ordinates of Q and R is constant, show that the locus of P is another parabola.

6. The tangent to the parabola $y^2 = 4ax$ at $P\,(at^2, 2at)$ cuts the axis in L and any line through L meets the parabola in points with parameters t_1, t_2. Prove that t_1, t, t_2 are in geometrical progression.

7. Show that the parabolas $x^2 + 4y = 4$, $x^2 + 2y = 2x$ touch one another, and find the equation of their common tangent. Find the coordinates of the vertex of the inner parabola.

8. The normal to the parabola $y^2 = 4ax$ at $P\,(at^2, 2at)$ meets the axis of the parabola at G and GP is produced beyond P to Q so that $GP = PQ$. Show that the locus of Q is the parabola $y^2 = 16a\,(x + 2a)$.

9. Show that the line $m\,(y - 2) = m^2\,(x - 1) + 1$ touches the parabola $y^2 - 4y = 4x - 8$ for all values of m other than zero.

Find the common tangent of this parabola and $y^2 = 4x$.

10. Show that the normal at $(at^2, 2at)$ to the parabola $y^2 = 4ax$ is given by $tx + y = at^3 + 2at$.

Prove that the locus of the point of intersection of the normals at the ends of a focal chord of the curve is the parabola $y^2 = a\,(x - 3a)$.

11. Show that the chord of the parabola $y^2 = 4ax$ which is bisected at the point (h, k) is given by

$$2ax - ky = 2ah - k^2.$$

Find the locus of the mid-point of a chord of the above parabola which touches the parabola $y^2 + 4ax = 0$.

12. Show that the normal at $P\,(at^2, 2at)$ to the parabola $y^2 = 4ax$ meets the curve again at the point Q whose parameter is $-(2 + t^2)/t$.

If A is the vertex and S the focus and if PR, parallel to AQ, meets the axis in R, show that $AR = 2SP$.

13. Find the point of intersection of the tangents at $(at^2, 2at)$, $(at'^2, 2at')$ to the parabola $y^2 = 4ax$.

If two perpendicular tangents meet at T and the corresponding normals meet at N, show that, for all such pairs of tangents, TN is parallel to the axis of the parabola and the locus of N is another parabola.

14. Prove that, if the chord joining two variable points P, Q on a parabola always passes through a fixed point on the axis, the locus of the point of intersection of the normals at P and Q is another parabola.

15. The normal at a point P of the parabola $y^2 = 4ax$ meets the curve again at Q and the tangents at P and Q intersect in R. Find the locus of the mid-point of PR.